

# Localization of maximal entropy random walk

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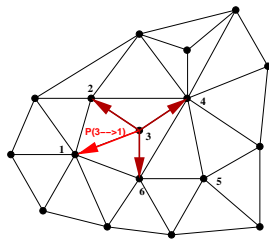
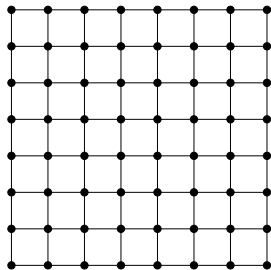
# Outline

- Introduction/motivation;
- Generic Random Walk;
- Entropy;
- Maximal Entropy Random Walk;
- Localization
- Summary and open problems

# Introduction

- Random walk on graphs;
- Local and global definitions;
- Global definition  $\longrightarrow$  Maximal Entropy Random Walk;

# RW on a finite connected graph



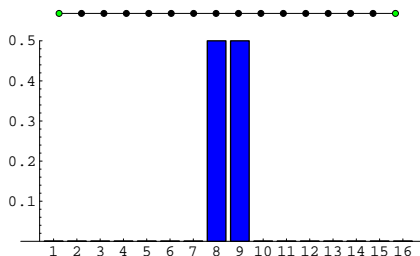
- Adjacency matrix:  $A_{ij} = 1(0)$  if  $ij$  are (not) neighbors;
- Node degree:  $k_i = \sum_j A_{ij}$ ;
- Hopping probability:  $P(i \rightarrow j) = P_{ij}$
- Stochastic matrix  $0 \leq P_{ij} \leq A_{ij}$ ;  $\sum_j P_{ij} = 1$ ;

# Occupation probability

- Master equation:  $\pi_i(t+1) = \sum_j \pi_j(t) P_{ji}$
- Stationary state:  $\pi_i^* = \sum_j \pi_j^* P_{ji}$
- Generic Random Walk:

$$P_{ij} = \frac{A_{ij}}{k_i} \longrightarrow \pi_i^* = \frac{k_i}{\sum_j k_j}$$

- Example:

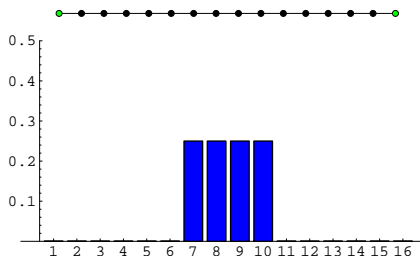


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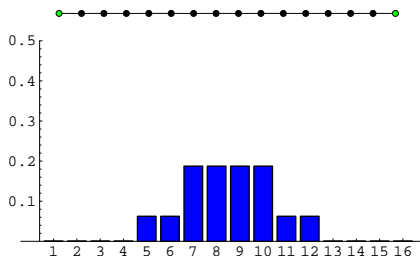


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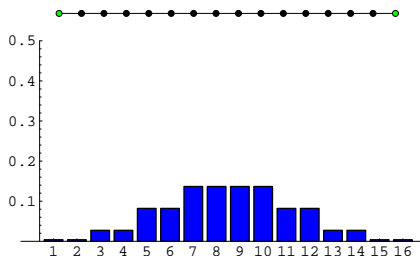


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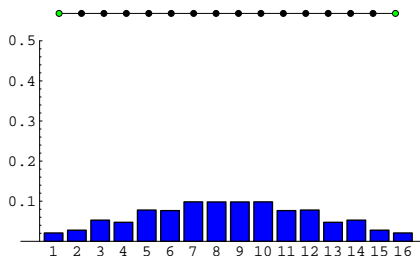


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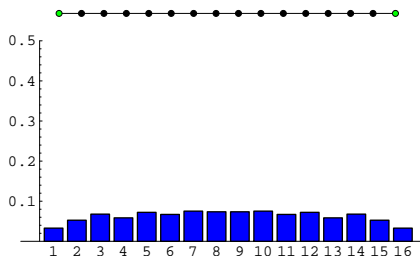


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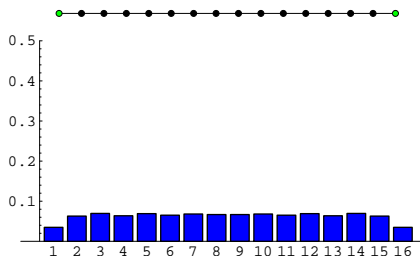


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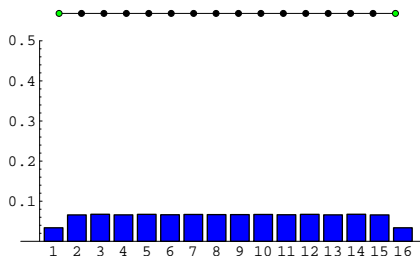


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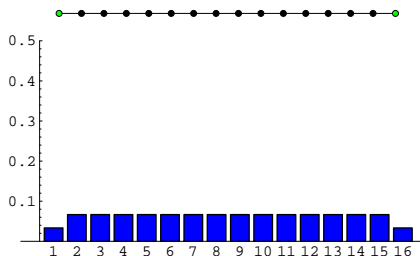


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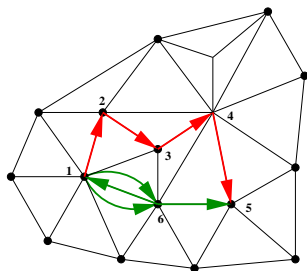
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- Example:



# Random trajectories $\gamma_{ab}^{(t)}$



- Probability:  $P(\gamma_{i_0 i_t}^{(t)}) = P_{i_0 i_1} P_{i_1 i_2} \dots P_{i_{t-1} i_t}$ ;
- GRW:  $P(\gamma_{i_0 i_t}^{(t)}) = (k_{i_0} k_{i_1} \dots k_{i_{t-1}})^{-1}$ ;
- Example:  
 $P(\gamma_{red}) = (7 \cdot 5 \cdot 4 \cdot 8)^{-1} = 1/1120$ ;  
 $P(\gamma_{green}) = (7 \cdot 6 \cdot 7 \cdot 6)^{-1} = 1/1764$ ;
- MERW:  $P(\gamma_{ab}^{(t)}) = F(t, a, b)$  ?

# Entropy of random trajectories

- $S_t = - \sum_{\{\gamma_{ab}^{(t)}\}} P(\gamma_{ab}^{(t)}) \ln P(\gamma_{ab}^{(t)})$
- Entropy production rate (Shannon, McMillan):

$$s \equiv \lim_{t \rightarrow \infty} \frac{S_t}{t} = - \sum_i \pi_i^* \sum_j P_{ij} \ln P_{ij}$$

- $s_{\text{GRW}} = \frac{\sum_i k_i \ln k_i}{\sum_i k_i} = \langle \ln k_i \rangle_*$
- Maximal entropy:

$$s_{\text{max}} = \frac{\ln N_t}{t} = \frac{\ln(A^t)_{ab}}{t} \sim \ln \lambda_{\text{max}}$$

- Inequality:

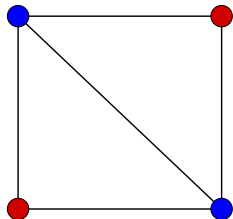
$$s_{\text{GRW}} \leq s_{\text{max}}$$

# Frobenius-Perron eigenvalue

- FP:  $k_{min} \leq \lambda_{max} \leq k_{max}$
- New inequality:

$$\exp\langle \ln k_i \rangle_* \leq \lambda_{max}$$

- Example:

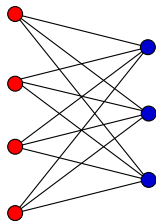
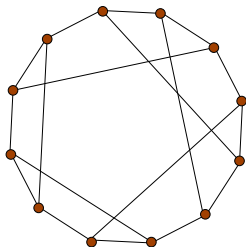


$$\exp\langle \ln k_i \rangle_* = 108^{1/5} \approx 2.55085;$$
$$\lambda_{max} = (1 + \sqrt{17})/2 \approx 2.56155;$$



# Equality $S_{GRW} = S_{max}$

- $\exp\langle \ln k_i \rangle_* = \lambda_{max}$ ;
- $k$ -regular graphs;  $\lambda_{max} = k$ ;
- bi-regular bipartite graphs:  $\lambda_{max} = \sqrt{k_1 k_2}$ ;



# MERW

- Let  $\sum_i \psi_i^2 = 1$  and

$$\sum_j A_{ij} \psi_j = \lambda_{\max} \psi_i$$

- Transition probability:

$$P_{ij} = \frac{A_{ij} \psi_j}{\lambda_{\max} \psi_i}$$

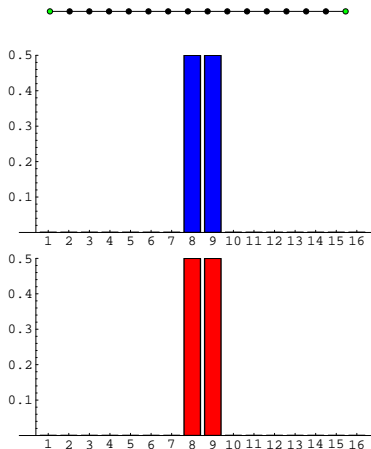
- Trajectories:

$$P(\gamma_{ab}^{(t)}) = \frac{1}{\lambda_{\max}^t} \frac{\psi_b}{\psi_a}$$

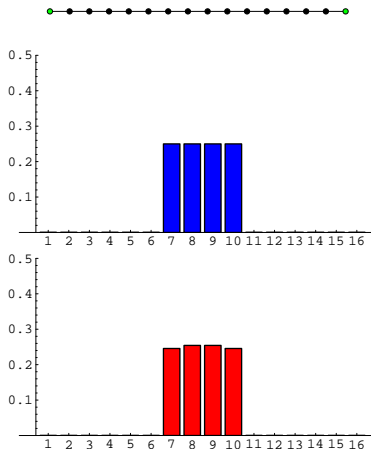
- Stationary distribution:  $\pi_i^* = \psi_i^2$

- Entropy rate:  $s_{\text{MERW}} = s_{\max} = \ln \lambda_{\max}$

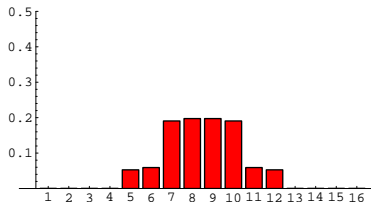
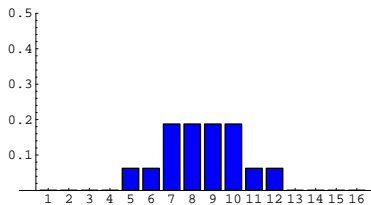
# GRW vs. MERW on a finite chain



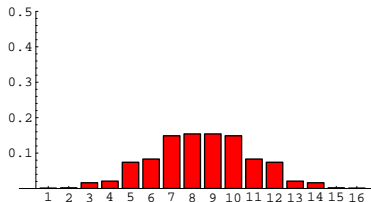
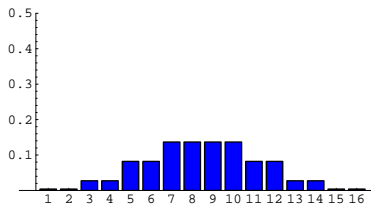
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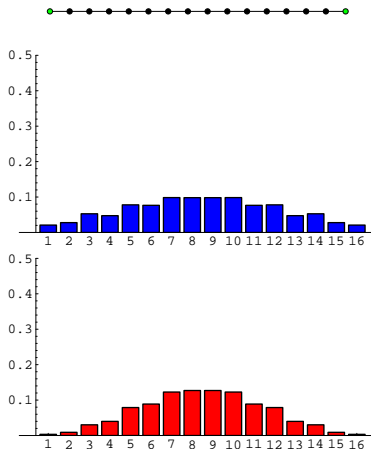
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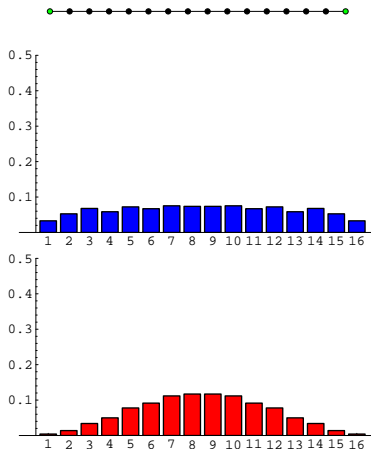
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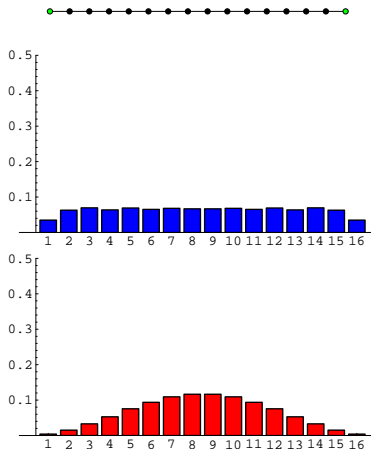


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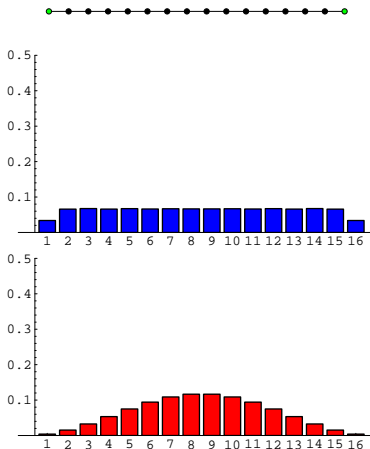




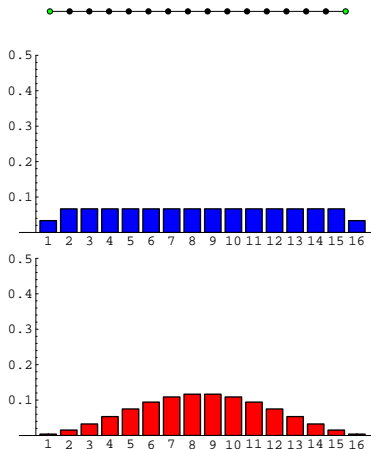
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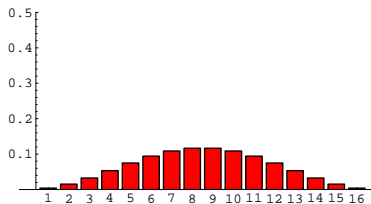
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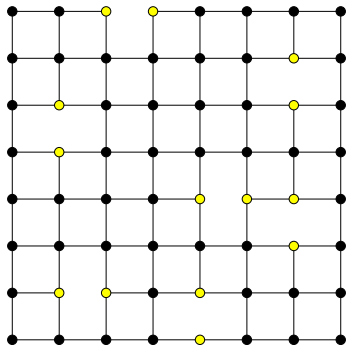


# Defects repulsion



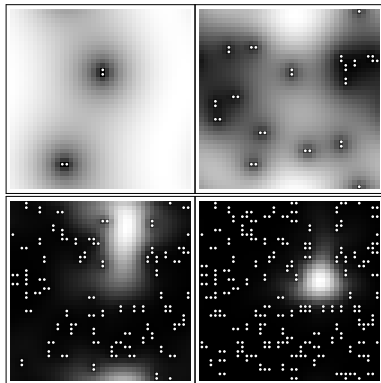
- $\pi_{*n} = \frac{2}{L+1} \sin^2 \frac{n\pi}{L+1}; \quad n = 1, \dots, L$
- Repulsion at the endpoints (defects)

# Lattice with dilution



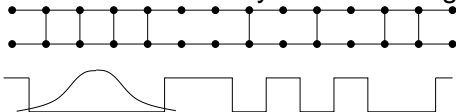
## MERW on lattice with dilution

- 2d square lattice + a small fraction  $q \ll 1$  of deleted links;
- Example:  $40 \times 40$ ;  $q = 0.001, 0.01, 0.05, 0.1$ :



# 1d example + Lifshitz argument

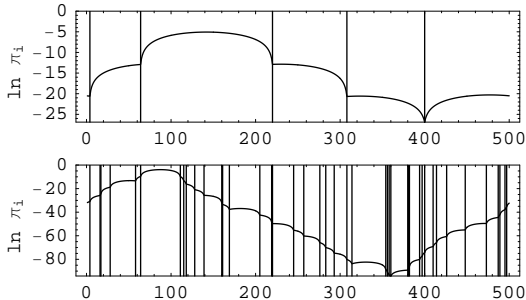
- ladder with randomly removed rungs:



- $\psi_{i+1} + \psi_{i-1} + r_i \psi_i = \lambda_{max} \psi_i$ ,  $r_i = 1$  with prob.  $p$  ;
- Lifshitz, Nieuwenhuizen, Luck
- $-(\Delta\psi)_i + v_i \psi_i = E_0 \psi_i$ ;  $v_i = 1 - r_i$ ;  $E_0 = k_{max} - \lambda_{max}$ ;
- localization on the longest chain of rungs  $l \sim \ln L / |\ln p|$
- $\psi_i \sim \sin i\pi/(l+1)$ ;  $E_0 \sim \pi^2/l^2$

# Numerical checks (1)

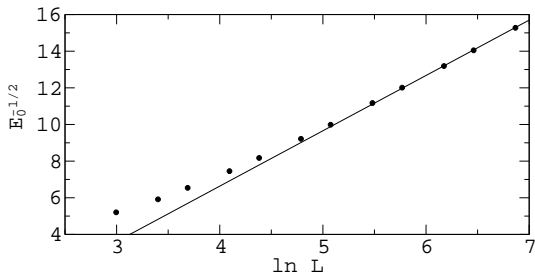
- $\pi_i^* = \psi_i^2$  for  $L = 500$ ;  $q = 1 - p = 0.01, 0.1$ .





## Numerical checks (2)

- $E_0 \approx (\pi |\ln p| / \ln L)^2$
- $L = 20, \dots, 960$ ;  $q = 1 - p = 0.1$



# Lifshitz spheres in $D > 1$

- $-(\Delta\psi)_i + v_i\psi_i = E_0\psi_i$
- $-(\Delta\psi)_i = E_0\psi_i$  with Dirichlet boundary condition;
- in the largest spherical region free of defects;
- in 2D: radius:  $R \approx (\ln L/(\pi|\ln p|))^{1/2}$

# Summary

- GRW maximizes local entropy;
- MERW maximizes global entropy;
- localization in the presence of weak disorder of the lattice;
- classical localization (but it is mapped into quantum problem)
- localization is related to Lifshitz states of a random operator;
- Inequality:

$$\lambda_{max} \geq \exp \left( \frac{\sum_i k_i \ln k_i}{\sum_i k_i} \right) \quad \text{where} \quad k_i = \sum_j A_{ij}$$

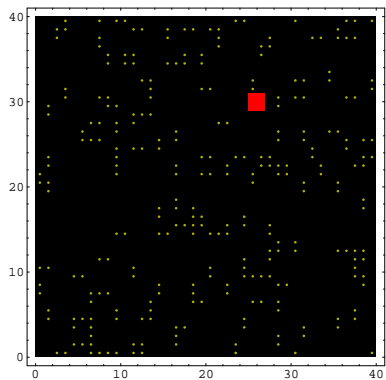
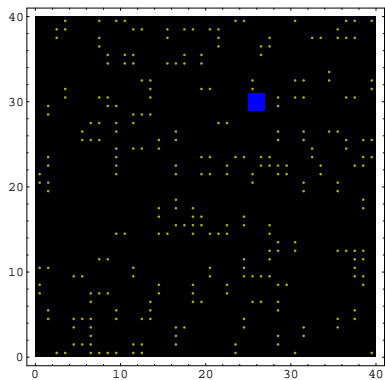
# Open problems

- Quantum amplitudes:

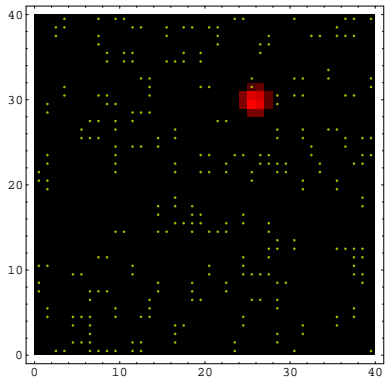
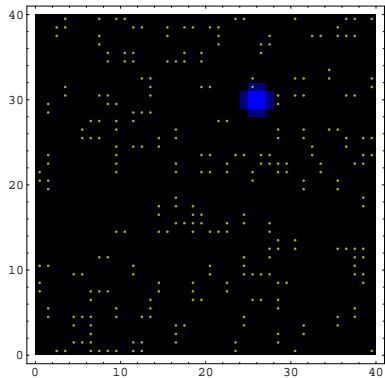
$$K_{ab} = \sum_t \sum_{\gamma_{ab}^{(t)}} e^{-S_E}; \quad S_E \sim t$$

- Dynamics: entropy barriers of local Lifshitz spheres;

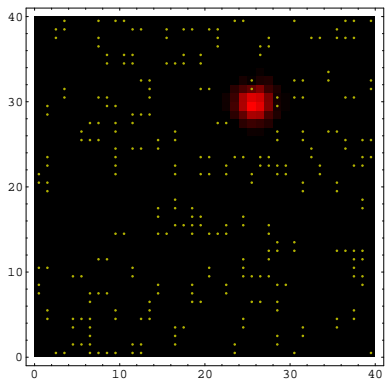
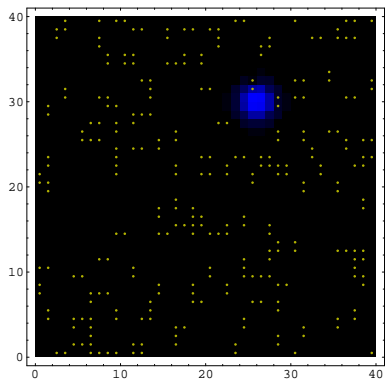
# MERW dynamics



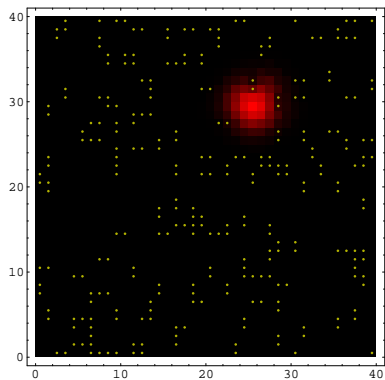
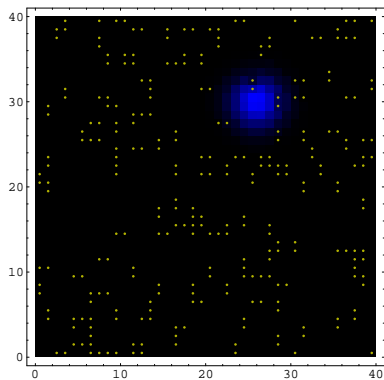
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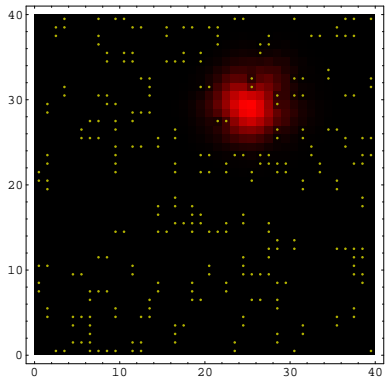
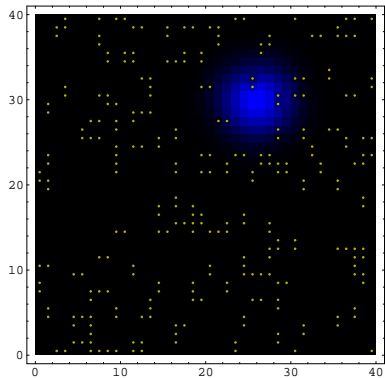


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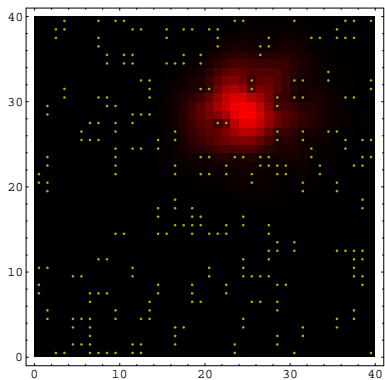
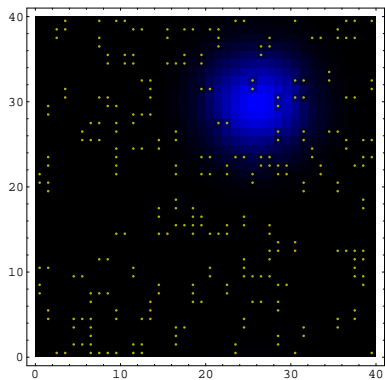




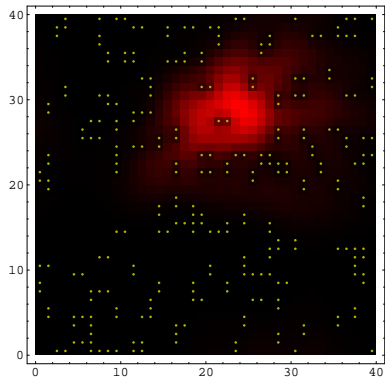
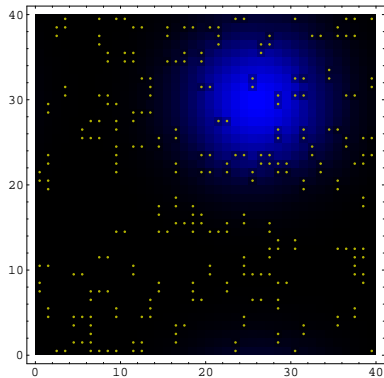
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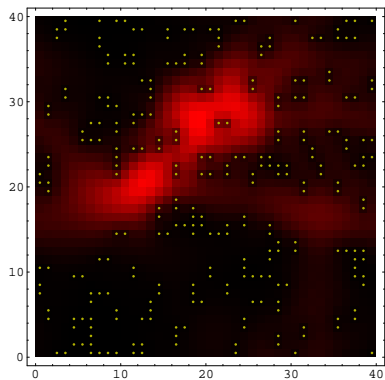
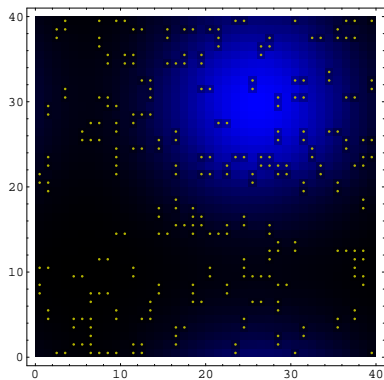
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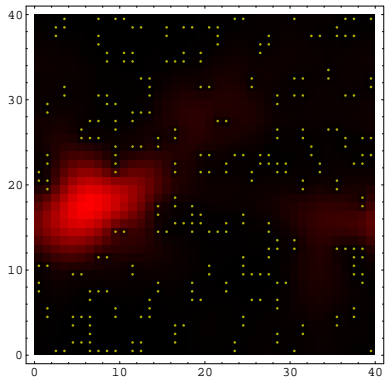
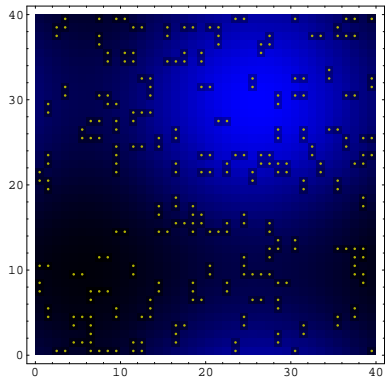
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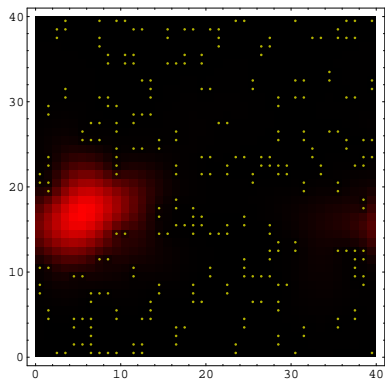
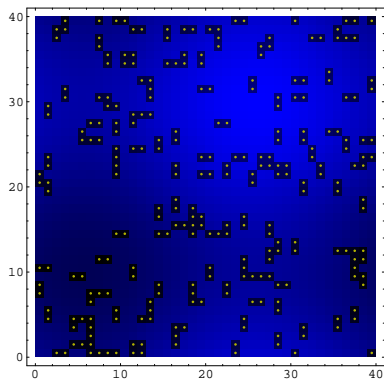
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